

Three Colorability of an Arrangement Graph of Great Circles

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Abstract

Stan Wagon asked the following in 2000. Is every zonohedron face 3-colorable when viewed as a planar map? An equivalent question, under a different guise, is the following: is the arrangement graph of great circles on the sphere always vertex 3-colorable?(The arrangement graph has a vertex for each intersection point, and an edge for each arc directly connecting two intersection points.) Assume that no three circles meet at a point, so that this arrangement graph is 4-regular.

In this note we have shown that all arrangement graphs defined as above are 3-colorable.

1 Introduction

One of the recent interesting graph coloring problem is the following: Is every zonohedron face 3-colorable when viewed as a planar map [Wag02]? An equivalent question, under a different guise, is the following: is the arrangement graph of great circles on the sphere always vertex 3-colorable [FHNS00] ? Here the arrangement graph has a vertex for each intersection point, and an edge for each arc directly connecting two intersection points. Assume that no three circles meet at a point, so that this arrangement graph is 4-regular. Koester has shown that arrangement graphs of circles in the plane or general circles on the sphere, can require four colors [Koe90]. All arrangement graphs of up to 11 great circles have been verified to be 3-colorable by O. Aichholzer.

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In this note we have shown that all arrangement graphs defined as above are 3-colorable. Our solution is based on the observation that all arrangement graphs of great circles contain two edge-disjoint identical (actually mirror image of each other) triangular (cycle of length three) closed chains.

2 The Triangular Chains

We assume that the sphere S and the great circles C_1, C_2, \dots, C_m are arranged so that one of the great circles say C_1 is also boundary at the horizon of S when it is seen as bird-eye view. For the sake of clarity in the figures we have color surface of all triangles on the back ground of the arrangement graph by dark-grey and color of all triangles on the fore ground of the arrangement graph on S by light-grey. We have the following simple observations:

(1) Let G be denote the graph corresponding to an arrangement of k great circles C_1, C_2, \dots, C_k . Then the number of triangles created by the arrangement of k great circles is $2k$.

(2) Let $T_i \in S_1$ be a fore-ground triangle of G then there exists mirror image back ground triangle $T'_i \in S'_1$ in G . For example for $k = 4$, in Fig. 1 mirror image back-ground triangles of the fore-ground triangles $(ABC), (CEF), (AGH), (BID)$ respectively are

$$(A'B'C'), (C'HI), (A'DE), (B'FD).$$

(3) In the arrangement of great circles graph G , the set of $k \geq 4$ triangles can be grouped into two sets as $S_1 = \{T_1, T_2, \dots, T_k\}$ and $S'_1 = \{T'_1, T'_2, \dots, T'_k\}$ such that each group of triangles forms a closed triangular edge-disjoint chains , where $S_1 \cap S'_1 = \emptyset$ and k denoting the number of the great circles.

Furthermore we have $V(T_i) \cap V(T_{i+1}) = v_i$ and $V(T'_i) \cap V(T'_{i+1}) = v'_i, i = 1, 2, \dots, k \pmod{k}$ for $k \geq 5$. For example for $k = 4$ in Fig. 1 these disjoint triangular chains respectively are $S_1 = \{(ABC), (CEF), (FB'G), (GHA)\}$ and $S'_1 = \{(A'B'C'), (C'HI), (IBD), (DEA')\}$.

(4) Since in any arrangement of great circles graph we do not allow more than two circles to meet at point, for fixed k all great circles graphs are isomorphic. This will permits us to consider one great circle graph for the given k .

3 Three-Coloring

Let us denote three colors by the numbers 1, 2, and 3. Our 3-coloring of G is based on the disjointedness of the two triangular chains. Depending on the parity of the number of great circles we have two cases to consider:

Case 1 *The number of great circles is even.*

Again consider the arrangement of four great circles shown in Fig.1. Consider the following two cycles: $c_1 = \{A', C', I, D\}$ and $c_2 = \{A, C, F, G\}$. Since these cycles are disjoint and of even length we can color their vertices with (1, 3)-(Kempe) chains. The remaining vertices in G are all the other non-adjacent vertices of the two triangular chains and therefore three coloring can be extended by coloring them with the color 2, i.e., the vertices B, E, B' , and H . When the number of great circles is even the above three coloring of arrangement of four circles can easily be generalized to any k even. For example in Fig. 2 we have shown three coloring of the arrangement of six great circles. Again the vertices other than the apex points of the triangles of the two disjoint triangular closed chains are colored by (1, 3)-chains. Then apex points of the triangles in the triangular chains form the *first*-level vertices of G and will be colored by the color 2. Similarly vertices (not yet colored) adjacent to the first level vertices that have been colored by 2 form the *second*-level vertices of G and we color them by the color 1. Actually the first and second level vertices can be viewed as the points of a rectangular-lattice L_r points embedded on the surface of the sphere S . For $k = 6$ the number of the lattice levels is two and it is enough to color these level-vertices respectively by colors 1 and 2 (see Fig. 2). In general for k great circles arrangement graph G the number of the rectangular lattice levels will be $k - 4$ and the vertices on the crossing points of the lattice are colored alternatingly with the colors 1 and 2. Note that, since the two triangular closed chains that have been colored by (1, 3)-chains, the overall coloring of G is a proper 3 coloring.

Case 2 *The number of great circles is odd.*

This case is different than the above since the length of the two disjoint triangular chains are odd. We will demonstrate our three coloring method for the arrangement graphs with $k = 5, 7$ (see Fig. 3,4) and explain the details for any odd k .

$k \equiv 0 \pmod{2}$: Consider the sequence of the colors in the two triangular chains in Fig.3, that is $T_1 = \{3^*, 1, 2, 1, 3, 2, 3^*\}$ and $T_2 = \{1^\#, 3, 2, 3, 1, 2, 1^\#\}$, where “ $*$ ” and “ $\#$ ” denote the color of the apex vertices of the light and dark-grey triangles. The coloring of the other vertices which are the apex points of the triangles is shown in Fig. 3.

$k \equiv 1 \pmod{2}$: Consider the sequence of the colors in the two triangular chains in Fig. 4, that is $T_1 = \{3^*, 2, 3, 2, 3, 1, 3, 1, 3^*\}$ and $T_2 =$

$\{3\#, 1, 3, 2, 3, 1, 3, 2, 3\#\}$, where “*” pointing the color of the apex-vertex in the light-grey triangle and “#” pointing the color of the apex vertex in the dark-grey triangle in the triangular chains of G . Starting these 3-colorings we can extend it to the other vertices of the graph as follows, where arrows show propagation of coloring on the vertices of G as:

$$T_1 = \begin{pmatrix} & & & & 3^* & & & \\ & & & & \diamond & \nwarrow 1 & & \\ & & & 2 \nearrow & \diamond & \nwarrow 3 \nearrow & & \\ & & 3 \nearrow & \diamond & \nwarrow 1 \nearrow & \diamond & \nwarrow 2 \nearrow & \\ & 1 \nearrow & \diamond & \nwarrow 2 \nearrow & \diamond & \nwarrow 3 \nearrow & \diamond & \nwarrow 3 \\ (3 \nearrow & \Delta & \nwarrow 2 \nearrow & \Delta & \nwarrow 3 \nearrow & \Delta & \nwarrow 1 \nearrow & \Delta \\ \uparrow & \uparrow \\ T_2 = (3 \downarrow & \nabla & \swarrow 2 \downarrow & \nabla & \swarrow 3 \downarrow & \nabla & \swarrow 1 \downarrow & \nabla \\ 1 \downarrow & \diamond & \swarrow 1 \downarrow & \diamond & \swarrow 2 \downarrow & \diamond & \swarrow 2 & \\ 2 \downarrow & \diamond & \swarrow 3 \downarrow & \diamond & \swarrow 1 & & & \\ 1 \downarrow & \diamond & \swarrow 2 & & & & & \\ 3\# & & & & & & & \end{pmatrix}$$

Symmetric of the coloring of the T_1 and T_2 have been shown by “↑” in the Fig. 4. Color the other vertices which are on the circle C_1 i.e., great circle at the horizon of sphere S , as $\{1, 2, 1, 3, 1, 2, 1, 2, 3, 2, 1, 2, 1\}$.

For the other odd values of k the sequence of colors of the triangular chains shown above can be generalized easily.

References

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Figure 1: Four great circles of arrangement graph.

Figure 2: Three coloring of the six-great circles graph.

Figure 3: Three coloring of the five-great circles graph.

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